Solutions to H.W. #2

1. Let $f: \mathbb{N} \mapsto \{1/n : n \geq 1\}$ be given by $f(n) = \frac{1}{n}$. Then clearly f is 1-1 and onto. Since $\{n\} = B_{1/2}^{\mathbb{N}}(n) = B_{1/2}^{\mathbb{R}}(n) \cap \mathbb{N} = (n-\frac{1}{2}, n+\frac{1}{2}) \cap \mathbb{N}$, we see that every subset of $\mathbb{N}$ is open in $\mathbb{N}$. Hence every function whose domain is $\mathbb{N}$ is continuous (why?). In particular, f is continuous.

Set $A = \{1/n : n \geq 1\}$. Then $f^{-1}: A \mapsto \mathbb{N}$ is continuous. In fact, if $1/n \in A$, letting $\delta = \frac{1}{n} - \frac{1}{n+1}$ yields $\{\frac{1}{n}\} = B_\delta^A(\frac{1}{n}) = (\frac{1}{n} - \delta, \frac{1}{n} + \delta) \cap A$, which proves that $\{\frac{1}{n}\}$ and hence every subset of A is open in A. That is, every function with domain A is continuous.

2. Let (M, d) be a metric space. Define $\rho: M \times M \mapsto [0, \infty)$ by $\rho(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$. Then ρ is a metric function. Hence (M, ρ) is a metric space. We now show that (M,d) and (M,ρ) are homeomorphic metric spaces by proving that d and ρ are equivalent metric functions. If $x_n \overset{d}{\mapsto} x$, then $d(x_n, x) \mapsto 0$ and therefore $\dfrac{d(x_n,x)}{1+d(x_n,x)} \mapsto 0$. Thus $x_n \overset{\rho}{\mapsto} x$ given that $x_n \overset{d}{\mapsto} x$. On the other hand, if $x_n \overset{\rho}{\mapsto} x$, then $\rho(x_n, x) \mapsto 0$

and we can write $d(x_n, x) = \dfrac{\rho(x_n, x)}{1-\rho(x_n, x)} \mapsto 0$.

Hence $d(x_n, x) \mapsto 0$ if and only if $\rho(x_n, x) \mapsto 0$ as desired.

Notice that the diameter of (M, ρ) is $\operatorname{diam}(M) =$
$= \sup\{\rho(x,y) : x, y \in M\} \leq \lim\limits_{d \mapsto \infty} \dfrac{d}{1+d} = 1$.

In particular, every metric space is homeomorphic to one of finite diameter.

3. Let $f : M \mapsto \mathbb{R}$ be given by $f(y) = d(y, x)$. Then f is continuous: $|f(y) - f(z)| = |d(y,x) - d(z,x)| \leq d(y, z)$. Hence f is Liepschitz.

By hypothesis, $f(x_n) \mapsto f(x)$ so $d(x_n, x) \mapsto d(x, x) = 0$
This proves that $x_n \overset{d}{\mapsto} x$.

4. If $A \subset B \subset M$, where B is totally bounded, then for $\epsilon > 0$ there are $x_1, \ldots, x_n \in B$ such that $B \subset \bigcup\limits_{i=1}^{n} B_\epsilon(x_i)$. But then $A \subset \bigcup\limits_{i=1}^{n} B_\epsilon(x_i)$, which shows that A can be covered by finitely many ϵ-balls.

5. Suppose that A is totally bounded. Then for any $\epsilon > 0$ there are points $x_1, \ldots, x_n \in A$ such that $A \subset \bigcup\limits_{i=1}^{n} B_{\epsilon/2}(x_i)$.
But then $A \subset \bigcup\limits_{i=1}^{n} C_{\epsilon/2}(x_i)$ where $C_{\epsilon/2}(x_i) = \{y \in M : d(x_i, y) \leq \epsilon/2\}$

are closed balls of diameter ϵ. Thus it is clear that every totally bounded set can be covered by finitely many closed sets of diameter at most ϵ.

On the other hand, if, for any $\epsilon > 0$, a set A can be covered by finitely many closed sets of diameter at most ϵ, it is clear from our class discussions and lemma 7.1 in the book that A must be totally bounded.

6. Suppose that $\bar{A}$ is totally bounded. Then $A \subset \bar{A}$ must also be totally bounded by the work done in problem 4. If, on the other hand, A is assumed to be totally bounded, then, by problem 5, every $\epsilon > 0$ corresponds to a cover of finitely many closed sets $F_1, \ldots, F_n$ with $\text{diam}(F_i) < \epsilon$. But $\bigcup_{i=1}^{n} F_i$ is a closed set that contains A. Hence $\bar{A} \subset \bigcup_{i=1}^{n} F_i$ (why?). Thus, it follows from problem 5 that $\bar{A}$ is totally bounded.

7. Suppose A is not totally bounded. Then there is some $\epsilon > 0$ for which no finite cover with ϵ-balls exists for A: let $x_1 \in A$, then $A \not\subset B_\epsilon(x_1)$. Select $x_2 \in A$ such that $x_2 \notin B_\epsilon(x_1)$. Now $A \not\subset B_\epsilon(x_1) \cup B_\epsilon(x_2)$. Pick $x_3 \in A \cap [B_\epsilon(x_1) \cup B_\epsilon(x_2)]^c \neq \emptyset$. Continue in this fashion to create a sequence $\{x_n\}_{n=1}^{\infty}$ such that $B = \{x_n : n \geq 1\} \subset A$ and $d(x_n, x_m) \geq \epsilon$ whenever $n \neq m$. Observe that $(B, \text{discrete})$ is homeomorphic to (B, d), because

the only Cauchy sequences that can be formed from the elements of (B,d) are eventually constant sequences (why?). Thus the d-metric is equivalent to the discrete metric.

8. Let $\{f_n\}$ be a sequence in ℓ_∞ defined by

$$f_n(k) = \begin{cases} 1 & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

Then the set $S = \{f_n : n \geq 1\} \subset \ell_\infty$ is closed and bounded. S is bounded because $\|f_n\|_\infty = 1$ for all n and closed because the only Cauchy sequences with range in S are the eventually constant sequences.

Notice that for $m \neq n$, $\|f_m - f_n\|_\infty = 1$. Hence S is not totally bounded.

9. Suppose that $A \subset M$ is complete. If $x \in M$ is a limit point of A, then there exists a sequence $\{x_n\} \subset A$ such that $x_n \xrightarrow{d} x$. Notice that the sequence $\{x_n\}$ is Cauchy in A. By hypothesis, this sequence must have a limit $y \in A$. But then $y \in M$ so $x_n \xrightarrow{d} y$ and $x_n \xrightarrow{d} x$, which means that $x = y$. In particular, $x \in A$. This proves that A is closed.

10. Let $\rho(x,y) = |\tan^{-1}x - \tan^{-1}y|$. Then $(\mathbb{R}, \rho)$ is not complete; The sequence $\{n\}_{n=1}^{\infty}$ is Cauchy in $(\mathbb{R}, \rho)$, yet it

fails to have a limit in $(\mathbb{R}, \rho)$:

$\{n\}_{n=1}^{\infty}$ is Cauchy, because $\rho(n,m) = |\tan^{-1}(n) - \tan^{-1}(m)| \leq$

$|\tan^{-1}(n) - \frac{\pi}{2}| + |\frac{\pi}{2} - \tan^{-1}(m)| \mapsto 0$ as $n, m \mapsto \infty$.

Notice that if $u \in \mathbb{R}$, $\lim\limits_{n\to\infty} \rho(n, u) = \lim\limits_{n\to\infty} |\tan^{-1}(n) - \tan^{-1}(u)| =$

$= |\frac{\pi}{2} - \tan^{-1}(u)| > 0$. Hence no real number u is the limit of the sequence $\{n\}_{n=1}^{\infty}$.

Let $\lambda(x,y) = |x^3 - y^3|$. Then $(\mathbb{R}, \lambda)$ is complete; Pick a Cauchy sequence $\{u_n\}_{n=1}^{\infty}$ in $(\mathbb{R}, \lambda)$. Then $\{u_n^3\}_{n=1}^{\infty}$ is a Cauchy sequence in $(\mathbb{R}$, usual$)$. Let $\lambda = \lim\limits_{n\to\infty} u_n^3$ be the limit of the real-valued sequence $\{u_n^3\}_{n=1}^{\infty}$. Then $\lambda = u^3$ for some $u \in \mathbb{R}$. Hence $u_n \overset{\lambda}{\mapsto} u$, so $\{u_n\}_{n=1}^{\infty}$ converges in $(\mathbb{R}, \lambda)$.

11. This is false! Let $(M, d) = (\mathbb{R}$, usual$)$ and (N, ρ) be $((-\frac{\pi}{2}, \frac{\pi}{2})$, usual$)$. Then $\tan^{-1}: \mathbb{R} \mapsto (\frac{-\pi}{2}, \frac{\pi}{2})$ is continuous, yet $(\frac{-\pi}{2}, \frac{\pi}{2})$ is not complete.

12. Let (M,d) and (N, ρ) be two metric spaces. Then $\sigma: (M \times N)^2 \mapsto \mathbb{R}$ given by $\sigma((a,b),(c,d)) = d(a,c) + \rho(b,d)$ defines a metric function on $M \times N$. Notice that a sequence $\{(a_n, b_n)\}_{n=1}^{\infty} \subset M \times N$ is Cauchy if and only if $\sigma((a_m, b_m), (a_n, b_n)) \mapsto 0$ whenever $n, m \mapsto \infty$. This happens

if and only if $d(a_m, a_n) \mapsto 0$ and $\rho(b_m, b_n) \mapsto 0$ as $n, m \mapsto \infty$.
Hence $\{(a_n, b_n)\}_{n=1}^{\infty}$ is Cauchy in $M \times N$ if and only if $\{a_n\}_{n=1}^{\infty}$ is Cauchy in M and $\{b_n\}_{n=1}^{\infty}$ is Cauchy in N.
Clearly, this implies that $M \times N$ is complete if and only if M and N are.

13. Let $\{f_n\}_{n=1}^{\infty} \subset S$ be defined by

$$f_n(k) = \begin{cases} \frac{1}{k} & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

In other words,
$f_1 = (1, 0, 0, 0, \ldots)$, $f_2 = (1, \frac{1}{2}, 0, 0, \ldots)$, $f_3 = (1, \frac{1}{2}, \frac{1}{3}, 0, \ldots)$,
$f_4 = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$, etc.
Then if $n > m$, $\|f_n - f_m\|_{\infty} = \frac{1}{m+1} \mapsto 0$ as $m \mapsto \infty$, which means that $\{f_n\}_{n=1}^{\infty}$ is Cauchy in S.
Notice that $f_n \mapsto f$ where $f(k) = \frac{1}{k}$ is the harmonic sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$. Clearly $f \notin S$ so S is not complete.

14. Let $\{f_n\}_{n=1}^{\infty}$ be Cauchy in c_0. Then, for each $n \in \mathbb{N}$, $\lim_{k \to \infty} f_n(k) = 0$. Since $c_0 \subset l_\infty$ and l_∞ is complete, we know that $f_n \mapsto f$, for some $f \in l_\infty$. Now

$$|f(k)| \leq |f(k) - f_n(k)| + |f_n(k)| \leq \|f - f_n\|_{\infty} + |f_n(k)|$$

Fix $n \in \mathbb{N}$ such that $\|f - f_n\|_\infty < \epsilon$ and let $k > K_0$ satisfy $|f_n(k)| < \epsilon$. Then,

$$|f(k)| \leq \|f - f_n\|_\infty + |f_n(k)| < \epsilon + \epsilon = 2\epsilon.$$

This establishes that $\lim_{k \to \infty} f(k) = 0$.